Lower bounds on the g-numbers of spheres without large missing faces

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Abstract

We establish several new lower bounds on the g-numbers of simplicial spheres without large missing faces. For this class of spheres, we derive bounds on the g-numbers in terms of the independence numbers of their graphs, extending a result of Chudnovsky and Nevo. As a consequence, we show that flag $(d-1)$-spheres—and more generally, flag normal $(d-1)$-pseudomanifolds—satisfy $g_2 \geq (1/2 - \delta(d))f_0$, where $\delta(d)$ is a function of d with $\delta(d) \to 0$ as $d \to \infty$. We further prove that, for simplicial $(d-1)$-spheres without large missing faces, an initial segment of the g-vector forms a level sequence, yielding additional inequalities among the g-numbers. Finally, we show that simplicial 4-spheres without missing faces of dimension greater than two satisfy $g_2 \geq \frac{2}{5}f_0 - \frac{6}{5}$.

1 Introduction

In this paper, we establish new lower bounds on the face numbers of simplicial spheres without large missing faces. Throughout, by a sphere we mean a (simplicial) $\mathbb{Z}/2\mathbb{Z}$-homology sphere. We denote by $f_i(\Delta)$ the number of i-faces of a simplicial complex Δ. The celebrated Lower Bound Theorem asserts that if Δ is a normal pseudomanifold of dimension $d-1 \geq 3$, then $f_1(\Delta) - df_0(\Delta) + \binom{d+1}{2} \geq 0$, and moreover, equality holds if and only if Δ is a stacked sphere [4, 17, 11, 34]. This theorem is usually phrased in terms of the g-numbers—certain alternating sums of the face numbers. In this language, the theorem states that for any normal pseudomanifold Δ of dimension at least three, $g_2(\Delta) \geq 0$, with equality if and only if Δ is a stacked sphere.

In the same spirit, the Generalized Lower Bound Theorem (namely, the linear inequalities of the g-theorem) asserts that if Δ is a $(d-1)$-sphere, then for all $1 \leq i \leq d/2$, we have $g_i(\Delta) \geq 0$ [31, 1, 2, 30, 18]. Furthermore, equality $g_i(\Delta) = 0$ for some $1 \leq i \leq d/2$ holds if and only if Δ is $(i-1)$-stacked; see [23, 24]. The notion of $(i-1)$-stackedness generalizes that of stackedness; instead of giving precise definitions, we merely note that a stacked sphere always has missing faces of dimension $\geq d-1$, while an $(i-1)$-stacked sphere has missing faces of dimension $\geq d-i+1$.

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This makes it plausible to conjecture that spheres without large missing faces should satisfy significantly stronger lower-bound–type results. Of particular interest are flag spheres—these are spheres without missing faces of dimension larger than one. A motivation for their study comes from their surprising metric properties first noticed by Gromov; see [15]. Building on Gromov's results, and motivated by the Hopf conjecture in Riemannian geometry, Charney and Davis [7] posited the following purely combinatorial conjecture:

Conjecture 1.1 (Charney–Davis Conjecture). *Let $k \geq 2$. Then all flag $(2k-1)$-spheres satisfy $g_k - g_{k-1} + \cdots + (-1)^k g_0 \geq 0$.*

At present, this conjecture is known to hold only in the case $k = 2$ [10], and it remains wide open in all higher-dimensional cases. Inspired by the Charney–Davis Conjecture and the Generalized Lower Bound Conjecture, Gal [12] defined the γ-numbers $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor d/2 \rfloor}$ of a $(d-1)$-sphere as certain weighted alternating sums of the h-numbers. Consequently, the γ-numbers can be expressed as linear combinations of the g-numbers. For instance, when $d = 2k$, one has $\gamma_k = g_k - g_{k-1} + \cdots + (-1)^k g_0$. Gal conjectured that the γ-numbers of *flag* spheres are all nonnegative. Very little progress has been made on Gal's conjecture. In a very recent paper of Chudnovsky and Nevo [8], it was shown that for flag $(d-1)$-spheres, $g_2 \geq \frac{1-\delta(d)}{2d+1} f_0$, where $\delta(d) \to 0$ as $d \to \infty$.

We denote by $S(j, d-1)$ the collection of ($\mathbb{Z}/2\mathbb{Z}$-homology) $(d-1)$-spheres without missing faces of dimension larger than j. In particular, $S(d, d-1)$ is the class of all $(d-1)$-spheres, while $S(1, d-1)$ is the class of all flag $(d-1)$-spheres. For the more general cases of spheres in $S(j, d-1)$ with $1 \leq j \leq d$, Nevo proposed a conjecture that interpolates between Gal's Conjecture for $S(1, d-1)$ and the generalized Lower Bound Theorem for $S(d, d-1)$; see [26, Conjecture 1.5]. For instance, for spheres in $S(2, 4)$, his conjecture states that $g_2 \geq g_1 = f_0 - 6$.

While we are still very far from proving Gal's or Nevo's conjectures, in this paper we use the affine stress spaces and the Stanley–Reisner rings to establish several new lower bounds on the g-numbers of spheres in $S(j, d-1)$. Our results can be summarized as follows:

1. We prove that any sphere in $S(2, 4)$ satisfies $g_2 \geq \frac{2}{5} f_0 - \frac{6}{5}$; see Theorem 4.7. This improves the bound given in [26, Lemma 4.2].

2. We establish a lower bound on the g_2-numbers of normal pseudomanifolds without large missing faces, and, more generally, lower bounds on the g-numbers of spheres without large missing faces in terms of the independence numbers of their graphs; see Theorems 5.1, 5.3, and 5.4. In particular, this implies that flag normal $(d-1)$-pseudomanifolds satisfy $g_2 \geq (1/2 - \delta(d)) f_0$, where $\delta(d)$ is a function of d with $\delta(d) \to 0$ as $d \to \infty$.

3. Using results on level rings, we obtain several additional inequalities on the g-numbers; see Corollary 6.5. As a consequence, we produce counterexamples to the conjectures on affine stress spaces raised in [28]; see Corollaries 6.9 and 6.11.

The structure of the paper is as follows. Section 2 reviews background on face enumeration for spheres and the relevant Stanley–Reisner ring theory. Section 3 discusses affine stress theory together with the cone lemma and some of its applications. Section 4 focuses on spheres in $S(2, 4)$, where we establish the lower bound in Theorem 4.7. In Section 5, we investigate the relationship between the g-numbers of spheres and the independence numbers of the associated graphs. Finally, Section 6 derives additional relations among the g-numbers using algebraic tools and presents counterexamples to two conjectures.

2 Preliminaries

2.1 Simplicial complexes and spheres

An (abstract) *simplicial complex* Δ with vertex set $V = V(\Delta)$ is a nonempty collection of subsets of V that is closed under inclusion and contains all singletons; that is, $\{v\} \in \Delta$ for all $v \in V$. Given a $(d+1)$-set V, the collection of all subsets of V is a *d-simplex*, which we usually denote by $\overline{V}$, or σ^d when we do not need to emphasize the vertex set. Similarly, the collection of all subsets of V except V itself is the *boundary complex* of a d-simplex, denoted by $\partial\overline{V}$ or $\partial\sigma^d$.

The elements of a simplicial complex Δ are called *faces* of Δ. A face τ of Δ has *dimension* i if $|\tau| = i+1$; in this case we say that τ is an *i-face*. We usually refer to 0-faces as *vertices*, 1-faces as *edges*, and the maximal under inclusion faces as *facets*. For brevity, we sometimes denote an i-face $\{a_1, a_2, \ldots, a_{i+1}\}$ by $a_1 a_2 \ldots a_{i+1}$. The *dimension of* Δ is $\max\{\dim \tau : \tau \in \Delta\}$. A set $\tau \subseteq V$ is a *missing face* of Δ if τ is not a face of Δ, but every proper subset of τ is a face of Δ; it is a *missing i-face* if $|\tau| = i+1$.

Let τ be a face of Δ. The *star* and *link* of τ are defined as

$$\mathrm{st}(\tau) = \mathrm{st}(\tau, \Delta) = \{\sigma \in \Delta \ : \ \sigma \cup \tau \in \Delta\} \quad \text{and} \quad \mathrm{lk}(\tau) = \mathrm{lk}(\tau, \Delta) = \{\sigma \in \mathrm{st}(\tau) \ : \ \sigma \cap \tau = \emptyset\}.$$

When $\tau = v$ is a vertex, we also define $\Delta \backslash v = \{\sigma \in \Delta : v \notin \sigma\}$; this subcomplex of Δ is called the *antistar* of v.

A subcomplex of Δ is called *induced* if it is of the form $\Delta[W] = \{\tau \in \Delta : \tau \subseteq W\}$ for some $W \subseteq V(\Delta)$. The subcomplex of Δ consisting of all faces of Δ of dimension $\leq k$ is called the *k-skeleton* of Δ and is denoted $\mathrm{Skel}_k(\Delta)$; the 1-skeleton of Δ is also known as the *graph* of Δ. Finally, if Δ and Γ are two simplicial complexes on disjoint vertex sets, then their *join* is

$$\Delta * \Gamma = \{\sigma \cup \tau : \sigma \in \Delta, \tau \in \Gamma\}.$$

If Γ is a 0-simplex, that is, $\Gamma = \{v, \emptyset\}$, we write $\Delta * \Gamma = \Delta * v$ and call this complex the *cone* over Δ with apex v. If Γ is $\partial\sigma^1$ (that is, the complex consisting of two disjoint vertices), we write $\Delta * \Gamma = \Sigma\Delta$ and call $\Sigma\Delta$ the *suspension* of Δ.

Let $\mathbb{F}$ be a field. A pure $(d-1)$-dimensional simplicial complex is an $\mathbb{F}$-*homology sphere* if, for every face σ (including the empty face), the link $\mathrm{lk}(\sigma)$ has the homology of a $(d-1-|\sigma|)$-dimensional sphere (over $\mathbb{F}$). Denote by $\|\Delta\|$ the geometric realization of Δ. We say that Δ is a *simplicial* $(d-1)$*-sphere* if $\|\Delta\|$ is homeomorphic to a $(d-1)$-dimensional sphere. It is known that a $\mathbb{Z}/2\mathbb{Z}$-homology sphere is always an $\mathbb{R}$-homology sphere; see [18, Lemma 2.1]. Moreover, a simplicial sphere is an $\mathbb{F}$-homology sphere for any field $\mathbb{F}$. The boundary complex of a simplicial polytope P, ∂P, is a simplicial $(d-1)$-sphere; such spheres are called *polytopal*.

In this paper, we work with the class of $\mathbb{Z}/2\mathbb{Z}$-homology spheres, which we refer to simply as spheres. We define $S(i, d-1)$ to be the set of $\mathbb{Z}/2\mathbb{Z}$-homology $(d-1)$-spheres whose missing faces all have dimension at most i. For example, if we set $K(i, d-1) := \partial\sigma^i * \partial\sigma^i * \partial\sigma^{d-2i}$ for $d/3 \leq i \leq d/2$, then $K(i, d-1) \in S(i, d-1)$. Moreover, the class of flag $(d-1)$-spheres is precisely $S(1, d-1)$. We also note that the link of a $(j-1)$-face of a sphere in $S(i, d-1)$ is an element of $S(i, d-j-1)$.

Let Δ be a $(d-1)$-sphere. Denote by $f_i = f_i(\Delta)$ the number of i-faces of Δ. The *f-vector* of Δ is $(f_{-1}, f_0, \ldots, f_{d-1})$, and the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is defined by the identity

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{j=0}^{d} h_i t^{d-i}.$$

The *h-polynomial* of Δ is $h(\Delta, t) = \sum_{i=0}^{d} h_i t^i$. For example, the h-vector of the boundary complex of a d-simplex is $(1, 1, \dots, 1)$, and consequently, $h(\partial\sigma^d, t) = 1 + t + \cdots + t^d$. The *g-vector* of Δ, $g(\Delta) = (g_0, g_1, \dots, g_{\lceil d/2 \rceil})$, is defined by letting $g_0 = 1$ and $g_j = h_j - h_{j-1}$ for $1 \leq j \leq \lceil d/2 \rceil$. The Dehn–Sommerville relations [19] assert that the h-vector of a $(d-1)$-sphere is symmetric: $h_i = h_{d-i}$ for all $0 \leq i \leq d$. In particular, it follows that if d is odd, then $g_{\lceil d/2 \rceil} = 0$.

We will also need the notion of a normal pseudomanifold, a class of simplicial complexes that significantly generalizes spheres. A simplicial complex Δ is a *normal* $(d-1)$*-pseudomanifold* if all facets have dimension $d-1$, every $(d-2)$-face is contained in exactly two facets, and the link of each face of dimension at most $d-3$ is connected. The f-numbers and h-numbers of a normal $(d-1)$-pseudomanifold are defined in the same way as for spheres; in particular, we will also be interested in $g_2 = h_2 - h_1 = f_1 - df_0 + \binom{d+1}{2}$.

2.2 Graphs and the independence number

Let $G = (V, E)$ be a graph. If the vertex set is $V = \{v_1, v_2, \dots, v_\ell\}$ and the edge set is $E = \{v_i v_{i+1} : 1 \leq i \leq \ell - 1\}$, then the graph is a *path*, which we usually denote by $(v_1, v_2, \dots, v_\ell)$. If $v_\ell = v_1$ (and all other vertices are distinct), we say that the graph is a *cycle*.

The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set in G. The independence number has been extensively studied in graph theory. One of the most important classical results in this area is the following; see, for instance, [3, Chapter 41].

Theorem 2.1 (Turán's theorem)**.** *Let G be a graph with f_0 vertices and f_1 edges. Then*

$$\alpha(G) \geq \frac{f_0}{2f_1/f_0 + 1}.$$

When Δ is a simplicial complex, we write $\alpha(\Delta)$ for the independence number of its graph.

2.3 The Stanley–Reisner ring theory

For a $(d-1)$-dimensional simplicial complex Δ with vertex set $V = V(\Delta)$, let $\mathbb{R}[X] = \mathbb{R}[x_v : v \in V]$ be the polynomial ring over $\mathbb{R}$ with one variable for each vertex of Δ. The *Stanley–Reisner ideal* of Δ is the ideal of $\mathbb{R}[X]$ generated by the monomials corresponding to the missing faces of Δ:

$$I_\Delta = (x_{j_1} x_{j_2} \cdots x_{j_k} : \{j_1, \dots, j_k\} \text{ is a missing face of } \Delta).$$

The *Stanley–Reisner ring* (or *face ring*) of Δ is the quotient $\mathbb{R}[\Delta] := \mathbb{R}[X]/I_\Delta$. This is a graded ring, and we write $\mathbb{R}[\Delta]_i$ for its ith graded component. (We will occasionally consider a subfield $\mathbb{F}$ of $\mathbb{R}$ and write $\mathbb{F}[\Delta] := \mathbb{F}[X]/I_\Delta$ for the Stanley–Reisner ring of Δ over $\mathbb{F}$. The discussion below applies in this setting as well.)

Assume that Δ is a $\mathbb{Z}/2\mathbb{Z}$-homology $(d-1)$-sphere. An embedding of Δ in $\mathbb{R}^d$ is a function $p : V(\Delta) \to \mathbb{R}^d$. In this paper we usually work with a *generic embedding*. That is, choose $\{a_{v,j} : v \in V, j \in [d]\} \subset \mathbb{R}$, where $[d] := \{1, 2, \dots, d\}$, to be a set that is *algebraically independent* over $\mathbb{Q}$ and define $p(v) = (a_{v,1}, \dots, a_{v,d})$ for each vertex $v \in V(\Delta)$. Occasionally, when Δ is realizable as the boundary complex of a simplicial d-polytope $P \subset \mathbb{R}^d$, we also consider a *natural embedding* of Δ. In this case, we take p to be the map assigning to each vertex $v \in \Delta$ the coordinates of the corresponding vertex of P.

Given an embedding p of Δ, define $\theta_j := \sum_{v\in V} p(v)_j x_v$ for $1 \le j \le d$. When p is a generic or natural embedding of Δ, the quotient ring $\mathbb{R}[\Delta]/(\theta_1, \dots, \theta_d)$ is a finite-dimensional $\mathbb{R}$-vector space. In this case, the sequence $\Theta = \Theta(p) = (\theta_1, \dots, \theta_d)$ is called a *linear system of parameters* (l.s.o.p.) and $\mathbb{R}[\Delta]/(\Theta)$ is called an *Artinian reduction* of $\mathbb{R}[\Delta]$.

Since Δ is a $(d-1)$-sphere, the ring $\mathbb{R}[\Delta]$ is Gorenstein. Consequently, any Artinian reduction of $\mathbb{R}[\Delta]$ is a Poincaré duality algebra, with $\dim_{\mathbb{R}} \left(\mathbb{R}[\Delta]/(\Theta)\right)_i = h_i(\Delta)$ for all $0 \le i \le d$. Furthermore, letting $c = \sum_{v\in V} x_v$, the Hard Lefschetz theorem for polytopes and spheres [31, 22, 1, 30, 2, 18] implies that $\mathbb{R}[\Delta]/(\Theta, c)$ is also a finitely generated standard graded algebra, with $\dim_{\mathbb{R}}(\mathbb{R}[\Delta]/(\Theta, c))_i = g_i(\Delta)$ for all $0 \le i \le \lceil d/2 \rceil$ and all other graded components of $\mathbb{R}[\Delta]/(\Theta, c)$ vanish.

Macaulay's theorem (see [32, Theorem II.2.2]) asserts that Hilbert functions of standard graded algebras are M-sequences. To define an M-sequence, note that for all positive integers n and i, there is a unique expression

$$n = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}, \quad \text{where } n_i > n_{i-1} > \cdots > n_j \ge j \ge 1.$$

Define $n^{\langle i \rangle} := \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}$ for $n > 0$, and set $0^{\langle 0 \rangle} := 0$. A sequence of nonnegative integers $(a_0, a_1, \dots, a_m)$ is called an *M-sequence* if $a_0 = 1$ and $0 \le a_{i+1} \le a_i^{\langle i \rangle}$ for all $1 \le i \le m-1$. A weaker, but often more convenient, property of an M-sequence $(a_0, a_1, \dots, a_m)$ is the following: writing $a_i = \binom{x_i}{i}$ for a real number x_i, one has $0 \le a_{i+1} \le \binom{x_i+1}{i+1}$ for all $1 \le i \le m-1$.

An immediate consequence of the Hard Lefschetz theorem and Macaulay's theorem is:

Theorem 2.2. (g-theorem) *Let Δ be a $(d-1)$-sphere. Then $(g_0(\Delta), g_1(\Delta), \dots, g_{\lfloor d/2 \rfloor}(\Delta))$ is an M-sequence.*

The part of the g-theorem asserting that the g-numbers are nonnegative, together with the characterization of spheres satisfying $g_i = 0$ for a given $i \le \lfloor d/2 \rfloor$ (see [23, 24]), is known as the Generalized Lower Bound Theorem. Recall that a $(d-1)$-sphere Δ is called $(i-1)$-*stacked* if there exists an $\mathbb{R}$-homology d-ball B such that $\partial B = \Delta$ and $\mathrm{Skel}_{d-i}(B) = \mathrm{Skel}_{d-i}(\Delta)$. (In other words, all interior faces of B have dimension at least $d - i + 1$.)

Theorem 2.3. *Let Δ be a $(d-1)$-sphere. Then $g_i(\Delta) \ge 0$ for all $1 \le i \le \lfloor \frac{d}{2} \rfloor$. Furthermore, $g_i(\Delta) = 0$ for some $1 \le i \le \lfloor \frac{d}{2} \rfloor$ if and only if Δ is $(i-1)$-stacked.*

2.4 Corollaries on the g-numbers and γ-numbers

Theorem 2.3 shows that a $(d-1)$-sphere with $g_i = 0$ for some $i \le \lfloor d/2 \rfloor$ does not belong to $S(d-i, d-1)$. This immediately implies the following.

Corollary 2.4. *If $i \le \lfloor d/2 \rfloor$ and $\Delta \in S(d-i, d-1)$, then $g_i(\Delta) \ge 1$.*

In the case $i = 2$, Nevo and Novinsky provided the following characterization of spheres in $S(d-2, d-1)$ with $g_2 = 1$; see [27].

Theorem 2.5. *Let $d \ge 4$. Suppose $\Delta \in S(d-2, d-1)$ satisfies $g_2(\Delta) = 1$. Then either $\Delta = \partial\sigma^i * \partial\sigma^{d-i}$ for some $2 \le i \le d-2$, or $\Delta = C * \partial\sigma^{d-2}$, where C is a cycle.*

McMullen's integral formula provides additional relations between the g-numbers of a pure simplicial complex and the g-numbers of its vertex links; see [21, p. 183] and [33, Proposition 2.3]. Recall that when d is odd, every $(d-1)$-sphere satisfies $g_{\lceil d/2 \rceil} = 0$.

Lemma 2.6. *Let Δ be a pure simplicial complex of dimension $d-1$. Then for every $0 \le k \le \lfloor \frac{d-1}{2} \rfloor$,*

$$\sum_{v \in V(\Delta)} g_k(\mathrm{lk}(v)) = (k+1)g_{k+1}(\Delta) + (d+1-k)g_k(\Delta).$$

Corollary 2.7. *Let $k \ge 1$ and let $\Delta \in S(k, 2k)$. Then $g_k(\Delta) \ge \frac{f_0(\Delta)}{k+2}$. In particular, if $\Delta \in S(2,4)$, then $g_2(\Delta) \ge \frac{1}{4} f_0(\Delta)$.*

Proof: The vertex links of Δ lie in $S(k, 2k-1)$. Hence, by Corollary 2.4, we have $g_k(\mathrm{lk}(v)) \ge 1$ for all $v \in \Delta$. Applying McMullen's formula and using the fact that $g_{k+1}(\Delta) = 0$, we obtain

$$g_k(\Delta) = \frac{1}{k+2} \sum_{v \in V(\Delta)} g_k(\mathrm{lk}(v)) \ge \frac{f_0(\Delta)}{k+2}.$$

□

Nevo [26] conjectured a significantly stronger lower bound on the g_2-numbers of spheres in $S(2,4)$.

Conjecture 2.8. *Let $\Delta \in S(2,4)$. Then $g_2(\Delta) \ge g_1(\Delta)$.*

We conclude this section with a discussion of the face numbers of flag spheres. In this setting, the γ-numbers may be viewed as analogs of the g-numbers for general spheres. In particular, the following analog of McMullen's formula holds for the γ-numbers. Although the proof follows standard techniques, we were unable to find this result explicitly stated in the literature, and so we include it here for completeness.

Let Δ be a $(d-1)$-sphere. The γ-numbers of Δ are defined by the relation

$$h(\Delta, t) = \sum_{i=0}^{d} h_i(\Delta) t^i = \sum_{k=0}^{\lfloor d/2 \rfloor} \gamma_k(\Delta) t^k (1+t)^{d-2k}.$$

For example, one easily checks that $\gamma_0 = 1$, $\gamma_1 = f_0 - 2d$, and $\gamma_2 = f_1 - (2d-3)f_0 + 2d(d-2)$. Recall that, under the standard $\mathbb{N}$-grading (where $\mathbb{N}$ denotes the set of all nonnegative integers), the Hilbert series[1] $F(\Delta, t)$ of the Stanley–Reisner ring $\mathbb{R}[\Delta]$ is given by $h(\Delta, t)/(1-t)^d$.

Lemma 2.9. *Let $\Delta \in S(d, d-1)$. Then for every $0 \le i \le \lfloor (d-1)/2 \rfloor$,*

$$\sum_{v \in V(\Delta)} \gamma_i(\mathrm{lk}(v)) = (i+1)\gamma_{i+1}(\Delta) + (2d-4i)\gamma_i(\Delta).$$

Proof: Assume that $V(\Delta) = [n]$. Consider first the $\mathbb{N}^n$-graded Hilbert series of $\mathbb{R}[\Delta]$. For $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, write $t_1^{a_1} \dots t_n^{a_n} = \mathbf{t}^a$ and $\deg \mathbf{t}^a = a$. Then

$$F(\Delta, t_1, \dots, t_n) = \sum_{a \in \mathbb{N}^n} \dim \mathbb{R}[\Delta]_a \mathbf{t}^a = \sum_{\sigma \in \Delta} \prod_{v \in \sigma} \frac{t_v}{1 - t_v}.$$

[1] For background on Hilbert series of Stanley–Reisner rings, see Sections II.1 and II.2 of Stanley's book [32].

Applying the differential operator $\partial_c = \sum_{v=1}^{n} \partial t_v$, we obtain:

$$\partial_c F(\Delta, t_1, \ldots, t_n) = \sum_{v=1}^{n} \frac{1}{(1-t_v)^2} \sum_{\tau \in \mathrm{lk}(v)} \prod_{j \in \tau} \frac{t_j}{1-t_j}.$$

Setting $t_1 = \cdots = t_n = t$, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\Delta, t) = \frac{1}{(1-t)^2} \sum_{v=1}^{n} \frac{h(\mathrm{lk}(v), t)}{(1-t)^{d-1}} = \sum_{v=1}^{n} \frac{\sum_{k=0}^{\lfloor (d-1)/2 \rfloor} \gamma_k(\mathrm{lk}(v)) t^k (1+t)^{d-1-2k}}{(1-t)^{d+1}}.$$

On the other hand, differentiating $F(\Delta, t) = \frac{h(\Delta,t)}{(1-t)^d} = \frac{\sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i(\Delta) t^i (1+t)^{d-2i}}{(1-t)^d}$ with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\Delta, t) = \frac{\sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i(\Delta) t^{i-1} (1+t)^{d-2i-1} (i + (2d-2i)t + it^2)}{(1-t)^{d+1}}.$$

Note that $i + (2d-2i)t + it^2 = (2d-4i)t + i(1+t)^2$. Comparing coefficients of $t^i(1+t)^{d-1-2i}$ in the two expressions for $\frac{\mathrm{d}}{\mathrm{d}t} F(\Delta, t)$, we conclude that $\sum_{v=1}^{n} \gamma_i(\mathrm{lk}(v)) = (i+1)\gamma_{i+1}(\Delta) + (2d-4i)\gamma_i(\Delta)$ for all $1 \leq i \leq \lfloor (d-1)/2 \rfloor$. □

By definition, a $2k$-sphere satisfies $\gamma_{k+1} = 0$. Together with Lemma 2.9, this yields the following result; see also [12, Corollary 2.2.2].

Corollary 2.10. *If $\gamma_k \geq 0$ holds for all flag $(2k-1)$-spheres, then $\gamma_k \geq 0$ holds for all flag $2k$-spheres. In particular, the Davis–Okun theorem [10] implies that $\gamma_2 \geq 0$ holds for all flag 4-spheres. Furthermore, all flag 5-spheres satisfy $3\gamma_3 + 4\gamma_2 \geq 0$.*

3 The stress theory

3.1 The affine stress spaces

Continuing with the notation of Section 2, let Δ be a simplicial complex of dimension $d-1$ and let $p : V = V(\Delta) \to \mathbb{R}^d$ be an embedding. Let $X = \{x_v : v \in V\}$, let $\mathbb{F} \subseteq \mathbb{R}$ be a field that contains all coordinates $\{p(v)_j : v \in V, j \in [d]\}$, and let $\mathbb{F}[X]$ be the corresponding polynomial ring.

For each $v \in V$, consider the differential operator $\partial_{x_v} := \frac{\partial}{\partial x_v}$ acting on $\mathbb{F}[X]$. More generally, if $\mu = x_{i_1} \cdots x_{i_s} \in \mathbb{F}[X]$ is a monomial, define $\partial_\mu : \mathbb{F}[X] \to \mathbb{F}[X]$ by $\rho \mapsto \partial_{x_{i_1}} \cdots \partial_{x_{i_s}} \rho$. If $\ell = \sum_{v \in V} \ell_v x_v$ is a linear form in $\mathbb{F}[X]$, define

$$\partial_\ell : \mathbb{F}[X] \to \mathbb{F}[X] \quad \text{by} \quad \rho \mapsto \sum_{v \in V} \ell_v \cdot \partial_{x_v} \rho = \sum_{v \in V} \ell_v \frac{\partial \rho}{\partial x_v}.$$

For a monomial $\mu \in \mathbb{F}[X]$, we define its *support* by $\mathrm{supp}(\mu) =: \{v \in V : x_v | \mu\}$.

Definition 3.1. A homogeneous polynomial $\lambda = \lambda(X) = \sum_\mu \lambda_\mu \mu \in \mathbb{F}[X]$ of degree k is called an *affine k-stress* on (Δ, p) if it satisfies the following conditions:

- Every nonzero term $\lambda_\mu \mu$ of λ is supported on a face of Δ; that is, $\mathrm{supp}(\mu) \in \Delta$.
- $\partial_{\theta_i} \lambda = 0$ for all $i = 1, \ldots, d$.

- $\partial_c\lambda = \sum_{v\in V} \partial_{x_v}\lambda = 0$.

We say that a face τ *participates* in a stress ω, if there is a nonzero monomial μ of ω with $\tau \subseteq \mathrm{supp}(\mu)$. The *support* of ω is the subcomplex of Δ consisting of all faces that participate in ω.

The set of affine k-stresses on Δ forms a vector space over $\mathbb{F}$, which we denote by $\mathcal{S}^a_k(\Delta, p; \mathbb{F})$. When the context is clear, we sometimes omit p and $\mathbb{F}$ from the notation. The stress space $\mathcal{S}^a_i(\Delta, p; \mathbb{F})$ coincides with a certain graded component of the Macaulay inverse system of $I_\Delta + (\Theta(p), c) \subseteq \mathbb{F}[X]$; see [25] for more details. In particular, the Hard Lefshetz theorem for polytopes and spheres admits the following (weaker) restatement in the language of stress spaces.

Theorem 3.2. *Let (Δ, p) be a $(d-1)$-sphere with a generic or natural embedding p. Then for every $1 \le k \le \lfloor d/2 \rfloor$, we have $\dim_{\mathbb{F}} \mathcal{S}^a_k(\Delta, p; \mathbb{F}) = g_k(\Delta)$. Here $\mathbb{F} \subseteq \mathbb{R}$ is any field containing $\mathbb{Q}(p(v)_j : v \in V(\Delta), j \in [d])$.*

If Λ is a full-dimensional subcomplex of Δ, then $\mathcal{S}^a_k(\Lambda, p) \subseteq \mathcal{S}^a_k(\Delta, p)$. In other words, any affine stress $\omega \in \mathcal{S}^a_k(\Lambda, p)$ is also an affine stress on (Δ, p) whose support is contained in Λ. Throughout the paper, we frequently apply this observation tacitly in the case $\Lambda = \mathrm{st}(\tau)$ for $\tau \in \Delta$.

3.2 The cone lemma and its applications

Many proofs in this paper rely on the following variant of the cone lemma from [29, Lemmas 3.1 and 3.2]. The proof is identical to that in [29], and so we omit it.

Lemma 3.3. *Let Δ be a $(d-2)$-dimensional simplicial complex with $V(\Delta) = [n]$, and let $\Gamma = 0 * \Delta$. Let $\{a_{u,j}, b_u : u \in [n], j \in [d-1]\} \subset \mathbb{R}$ be a set of numbers that are algebraically independent over $\mathbb{Q}$. Define $\mathbb{F}' = \mathbb{Q}\big(\frac{a_{u,j}}{b_u} : u \in [n], j \in [d-1]\big)$ and $\mathbb{F} = \mathbb{Q}(a_{u,j}, b_u : u \in [n], j \in [d-1]) = F'(b_1, \ldots, b_n)$. Embed Δ and Γ in $(\mathbb{F}')^{d-1}$ and $\mathbb{F}^d$, respectively, using maps p' and p defined by*

$$p'(u) = \left(\frac{a_{u,1}}{b_u}, \ldots, \frac{a_{u,d-1}}{b_u}\right), \quad p(u) = (a_{u,1}, ..., a_{u,d-1}, b_u) \text{ for all } u \in [n],$$

and set $p(0) = (0, \ldots, 0)$. Then every $\omega' \in \mathcal{S}^a_i(\Delta, p'; \mathbb{F}')$ lifts to an element $\omega \in \mathcal{S}^a_i(\Gamma, p; \mathbb{F})$. Moreover, $\mathrm{supp}(\omega) = \mathrm{Skel}_{i-1}(0 * \mathrm{supp}(\omega'))$.

In particular, we will frequently use the following corollary.

Corollary 3.4. *Let Σ be a $(d-1)$-sphere, let p be a generic embedding of Σ in $\mathbb{R}^d$, let $1 \le k < d$, and let $i \le \frac{d-k}{2}$. Then for every $(k-1)$-face τ of Σ with $g_i(\mathrm{lk}(\tau)) \ge 1$, there exists $\omega \in \mathcal{S}^a_i(\mathrm{st}(\tau), p; \mathbb{F})$ such that $\mathrm{supp}(\omega) \supseteq \mathrm{Skel}_{i-1}(\overline{\tau})$. Here $\mathbb{F} \subseteq \mathbb{R}$ is any field containing $\mathbb{Q}(p(v)_j : v \in V(\Delta), j \in [d])$.*

Proof: Throughout the proof, we set $a_{u,j} := p(u)_j$ for $u \in V(\Sigma), j \in [d]$.

The proof is by induction on k. When $k = 1$, $\tau = v$ is a vertex. Consider

$$\mathbb{F}' = \mathbb{Q}\left(\frac{a_{u,j} - a_{v,j}}{a_{u,d} - a_{v,d}} : u \in V(\mathrm{lk}(v), j \in [d-1])\right),$$

and define maps $q' : V(\mathrm{lk}(v)) \to (\mathbb{F}')^{d-1}$ and $q : V(\mathrm{st}(v)) \to \mathbb{F}^d$ by

$$q'(u) = \left(\frac{a_{u,1} - a_{v,1}}{a_{u,d} - a_{v,d}}, \ldots, \frac{a_{u,d-1} - a_{v,d-1}}{a_{u,d} - a_{v,d}}\right) \; \forall u \in V(\mathrm{lk}(v)), \; q(u) = p(u) - p(v) \; \forall u \in V(\mathrm{st}(v)). \quad (3.1)$$

Since the set $\{a_{u,i} : u \in \mathrm{st}(v), i \in [d]\}$ is algebraically independent, so is $\{a_{u,i} - a_{v,i} : u \in \mathrm{lk}(v), i \in [d]\}$. Because $g_i(\mathrm{lk}(v))) \geq 1$, there exists a nonzero $\omega' \in \mathcal{S}_i^a(\mathrm{lk}(v), q'; \mathbb{F}')$. By Lemma 3.3, the stress ω' lifts to a stress $\omega \in \mathcal{S}_i^a(\mathrm{st}(v), q; \mathbb{F})$ with v in its support. Since translations do not affect the space of affine stresses, we have $\mathcal{S}_i^a(\mathrm{st}(v), q; \mathbb{F}) = \mathcal{S}_i^a(\mathrm{st}(v), p; \mathbb{F})$, and the result follows.

Now assume $k > 1$ and τ is a $(k-1)$-face with $g_i(\mathrm{lk}(\tau)) \geq 1$. Let v be a vertex of τ, and set $\tau' = \tau \backslash v$. Then $\mathrm{lk}(v)$ is a $(d-2)$-sphere, τ' is a face of $\mathrm{lk}(v)$, and $|\tau'| = k-1$. Moreover, $\mathrm{lk}(\tau', \mathrm{lk}(v)) = \mathrm{lk}(\tau)$, so $g_i(\mathrm{lk}(\tau', \mathrm{lk}(v))) \geq 1$. Applying the inductive hypothesis to the face τ' in the sphere $(\mathrm{lk}(v), q')$, with q' defined by equation (3.1), we obtain a stress $\omega' \in \mathcal{S}_i^a(\mathrm{lk}(v), q')$ whose support is contained in $\mathrm{st}(\tau', \mathrm{lk}(v))$ and contains $\mathrm{Skel}_{i-1}(\overline{\tau'})$. Since $\mathrm{st}(\tau, \mathrm{st}(v)) = v * \mathrm{st}(\tau', \mathrm{lk}(v))$, Lemma 3.3 implies that ω' lifts to a stress $\omega \in \mathcal{S}_i^a(\mathrm{st}(v), q)$ whose support is contained in $\mathrm{st}(\tau)$ and contains $\mathrm{Skel}_{i-1}(v * \overline{\tau'}) = \mathrm{Skel}_{i-1}(\overline{\tau})$. This completes the proof. □

Remark 3.5. The only properties of spheres used in the proof of Corollary 3.4 are: (i) the link of a $(k-1)$-face of a homology $(d-1)$-sphere Δ is a homology $(d-k-1)$-sphere, and (ii) under a generic embedding p' of Σ in $\mathbb{R}^{d'}$, the dimension of the space $\mathcal{S}_j^a(\Delta, p')$ is $g_j(\Delta)$ for all $j \leq d/2$.

If Δ is a normal pseudomanifold, then the link of any face of Δ is again a normal pseudomanifold. By [11] and [34], property (ii) continues to hold when $j = 2$ and Δ is a normal pseudomanifold of dimension ≥ 4. Therefore, Corollary 3.4 also holds when $i = 2$ and Σ is a normal pseudomanifold. We will use this fact in the proof of Theorem 5.1.

We close this section with another application of Lemma 3.3, which will be used in Section 4 and further generalized in Section 5. Recall that the independence number of the graph of a simplicial complex Δ is denoted by $\alpha(\Delta)$.

Proposition 3.6. *Let $i \leq \frac{d-1}{2}$ and $j \geq i+1$, and let $\Delta \in S(d-j, d-1)$. Then $g_i(\Delta) \geq \alpha(\Delta)$. In particular, if $\Delta \in S(\lfloor \frac{d}{2} \rfloor, d-1)$, then $g_i(\Delta) \geq \alpha(\Delta)$ for all $i \leq \frac{d-1}{2}$.*

This result and its proof are motivated by the following recent result of Chudnovsky and Nevo [8]:

Theorem 3.7. *Let $d \geq 5$ and let Δ be a $(d-1)$-sphere. If every vertex link of Δ satisfies $g_2 \geq 1$, then $g_2(\Delta) \geq \alpha(\Delta)$. In particular, for $d \geq 5$ and $\Delta \in S(1, d-1)$, $g_2(\Delta) \geq \frac{1-\delta(d)}{2d+1} f_0(\Delta)$, where $\delta(d) \to 0$ as $d \to \infty$.*

Remark 3.8. Although [8] states the inequality $g_2(\Delta) \geq |I|$ for flag spheres, the proof there relies only on the weaker assumption that every vertex link satisfies $g_2 \geq 1$. Consequently, the lower bound on $g_2(\Delta)$ given in the theorem continues to hold for spheres in $S(d-3, d-1)$ and, similarly to Remark 3.5, even for normal $(d-1)$ pseufomanifolds whose missing faces all have dimension $\leq d-3$. (Here $d \geq 5$.)

Proof of Proposition 3.6: Consider a generic embedding p of Δ. let I be a maximum independent set in the graph of Δ; that is, $|I| = \alpha(\Delta)$. Since $\Delta \in S(d-j, d-1)$, all vertex links of Δ lie in $S(d-j, d-2)$. Because $j \geq i+1$ and $i \leq (d-1)/2$, it follows from Corollary 2.4 that $g_i(\mathrm{lk}(v)) \geq 1$ for every vertex $v \in \Delta$. Therefore, by Corollary 3.4, for each $v \in I$ there exists a stress $\omega_v \in \mathcal{S}_i^a(\mathrm{st}(v), p) \subseteq \mathcal{S}_i^a(\Delta, p)$ whose support contains v.

Now let $u, v \in I$ with $u \neq v$. Since I is an independent set, $v \notin \mathrm{st}(u)$, and hence v does not appear in the support of ω_u. Thus each ω_v has a vertex in its support that does not appear in the support of any other ω_u with $u \in I$, $u \neq v$. Consequently, the collection $\{\omega_v : v \in I\}$ is linearly independent. Therefore, $g_i = \dim \mathcal{S}_i^a(\Delta) \geq |I| = \alpha(\Delta)$. This completes the proof. □

4 The g_2-numbers of spheres in $S(2,4)$

With Lemma 2.6 and Proposition 3.6 at our disposal, we are now ready to investigate lower bounds on g_2 for spheres in $S(2,4)$. For this discussion, recall that $K(2,4) = \partial\sigma^2 * \partial\sigma^2 * \partial\sigma^1$ is a sphere in $S(2,4)$. The significance of $K(2,4)$ is that any $\Delta \in S(2,4)$ satisfies $f_0(\Delta) \geq f_0(K(2,4)) = 8$ [26, Theorem 1.1(b)];[2] moreover, $K(2,4)$ is the unique 8-vertex element of $S(2,4)$.

Also, as mentioned in Section 2.4 (see Corollary 2.7), a sphere $\Delta \in S(2,4)$ satisfies $g_2(\Delta) \geq f_0(\Delta)/4$, while it is conjectured that the much stronger inequality $g_2(\Delta) \geq g_1(\Delta) = f_0(\Delta) - 6$ should hold. Meanwhile, since $f_0(\Delta) \geq 8$, for any λ with $\frac{1}{4} \leq \lambda \leq 1$ we have

$$\frac{1}{4} f_0(\Delta) \leq \lambda f_0(\Delta) - (8\lambda - 2) \leq f_0(\Delta) - 6,$$

and all these inequalities hold as equalities for $\Delta = K(2,4)$.

The goal of this section is to determine the optimal value of λ for which we can prove that $g_2 \geq \lambda f_0 - (8\lambda - 2)$ for all spheres in $S(2,4)$. We will see in Theorem 4.7 that $\lambda = 2/5$ works.

4.1 Reduction

Given a simplicial complex Δ and an edge uv that is not contained in any missing face (equivalently, $\mathrm{lk}(uv) = \mathrm{lk}(u) \cap \mathrm{lk}(v)$), one can perform an operation that contracts uv. This operation replaces the vertices u and v with a new vertex u', and replaces every face F that intersects $\{u, v\}$ with $(F \backslash \{u, v\}) \cup \{u'\}$. The resulting complex Δ' is still a simplicial complex; moreover, if Δ is a homology $(d-1)$-sphere, then so is Δ'; see [27, Proposition 2.3]. For a sphere in $\Delta \in S(2,4)$ and an edge e, we say that contracting e is an *admissible contraction* if performing it yields a complex Δ' that still belongs to $S(2,4)$.

In what follows, we consider a *reduced* sphere in $S(2,4)$. We say that $\Delta \in S(2,4)$ is reduced if the following conditions hold:

1. Δ admits no admissible edge contractions.
2. There is no induced subcomplex $\Gamma \cong \partial\sigma^2 * \partial\sigma^2 \subseteq \Delta$, such that each connected component of $\Delta \backslash \Gamma$ contains at least two vertices. (Here $\Delta \backslash \Gamma$ denotes the induced subcomplex of Δ on the vertex set complementary to that of Γ.)

To see that it suffices to consider reduced complexes, suppose that $\Delta' \in S(2,4)$ is obtained from Δ by an admissible contraction of an edge e. Then $g_2(\Delta') = g_2(\Delta) - g_1(\mathrm{lk}(e))$. Since $\mathrm{lk}(e) \in S(2,2)$, we have $g_1(\mathrm{lk}(e)) \geq 1$. Recall also that $\lambda \leq 1$. Thus, if $g_2(\Delta') \geq \lambda f_0(\Delta') - (8\lambda - 2)$, then $g_2(\Delta) \geq \lambda f_0(\Delta') - (8\lambda - 2) + g_1(\mathrm{lk}(e)) \geq \lambda f_0(\Delta) - (8\lambda - 2)$, since $f_0(\Delta) = f_0(\Delta') + 1$ and $g_1(\mathrm{lk}(e)) \geq 1 \geq \lambda$. In other words, in this case, Δ also satisfies $g_2(\Delta) \geq \lambda f_0(\Delta) - (8\lambda - 2)$.

Similarly, suppose that Δ violates condition 2. Let Δ'_1 and Δ'_2 be the two 4-balls such that $\partial\Delta'_1 = \partial\Delta'_2 = \Gamma$ and $\Delta'_1 \cup_\Gamma \Delta'_2 = \Delta$. For $i = 1, 2$, define $\Delta_i = \Delta'_i \cup (\Gamma * v_i)$, where v_1 and v_2 are new vertices. Then $\Delta_i \in S(2,4)$ and $f_0(\Delta_i) < f_0(\Delta)$ for each i. Noting that $\Sigma\Gamma = K(2,4)$, and hence $g_2(\Sigma\Gamma) = 2 = \lambda f_0(\Sigma\Gamma) - (8\lambda - 2)$, we conclude that if $g_2(\Delta_i) \geq \lambda f_0(\Delta_i) - (8\lambda - 2)$, then

$$\begin{aligned} g_2(\Delta) &= g_2(\Delta_1) + g_2(\Delta_2) - g_2(\Sigma\Gamma) \\ &\geq \left(\sum_{i=1,2} (\lambda f_0(\Delta_i) - (8\lambda - 2))\right) - (\lambda f_0(\Sigma\Gamma) - (8\lambda - 2)) = \lambda f_0(\Delta) - (8\lambda - 2). \end{aligned}$$

[2] In fact, within the class $S(2,4)$, the sphere $K(2,4)$ simultaneously minimizes all face numbers [14, Theorem 3.1].

Thus, once again Δ satisfies $g_2(\Delta) \geq \lambda f_0(\Delta) - (8\lambda - 2)$.

This discussion implies the following result:

Proposition 4.1. *If, for some constant $1/4 \leq \lambda \leq 1$, all reduced spheres in $S(2,4)$ satisfy $g_2 \geq \lambda f_0 - (8\lambda - 2)$, then all spheres in $S(2,4)$ satisfy this inequality.*

The best scenario we can now hope for is the following: a reduced $\Delta \in S(2,4)$ is either $K(2,4)$, or it admits no admissible edge contraction and, consequently, contains many missing 2-faces. In the former case, we have $g_2 = g_1$. In the latter case, one expects g_2 to be strictly larger than the proposed lower bound. This situation may be compared with the behavior of the γ_2-numbers of flag 3-spheres: Venturello [35] constructed a flag 3-sphere with 12 vertices such that 1) it is not a suspension, 2) it admits no admissible edge contractions, and 3) except for the vertex links, it has no induced subcomplexes that are 2-spheres. The γ_2-number of this sphere is not zero; in fact, it equals 1.

4.2 A new lower bound on g_2

Throughout, we assume that $\Delta \in S(2,4)$ is reduced and that $f_0(\Delta) > 8$. Our strategy is to derive a lower bound on the number of vertices of Δ whose links satisfy $g_2 \geq 2$, and then use this bound, together with McMullen's formula, to establish a new lower bound on $g_2(\Delta)$. Specifically, our first goal is to prove the following.

Proposition 4.2. *Let $v \in \Delta$ be a vertex with $g_2(\mathrm{lk}(v)) = 1$. Then every neighbor u of v satisfies $g_2(\mathrm{lk}(u)) \geq 2$. In particular, the set of vertices v with $g_2(\mathrm{lk}(v)) = 1$ forms an independent set in the graph of Δ.*

In what follows, let $v \in V(\Delta)$ be a vertex with $g_2(\mathrm{lk}(v)) = 1$. Since $\mathrm{lk}(v) \in S(2,3)$, Theorem 2.5 implies that $\mathrm{lk}(v)$ is the join of two cycles $C = (u_1, u_2, \ldots, u_n, u_1)$ and $C' = (w_1, w_2, w_3, w_1)$, where $n \geq 3$. We use this notation in the next three lemmas.

Lemma 4.3. *If $n > 3$ and $g_2(\mathrm{lk}(u_2)) = 1$, then the edge vu_2 is contained in a missing 2-face of the form vu_2u_j. Furthermore, the complex $C' * (v, u_2, u_j, v)$ is an induced subcomplex of Δ.*

Proof: Since $\Delta \in S(2,4)$, if $e = vu_2$ is not contained in any missing 2-face, then e is not contained in any missing face at all. Hence, we may contract e to a new vertex v'. Denote the resulting complex by Δ'. Then Δ' is a 4-sphere. However, by our assumption that Δ is reduced, Δ' does not belong to $S(2,4)$, and therefore, it must have a missing 3-face F. This face must contain the new vertex v'; in other words, F must be of the form $\{v', x, y, z\}$. For this to occur, the subcomplex of Δ induced by the vertex set $\{v, u_2, x, y, z\}$ must be a 2-sphere $S \cong \partial\sigma^2 * \partial\sigma^1$, and moreover the edge vu_2 must belong to S. Since the only missing 2-face in the link of v is C', and since vu_2 is an edge of S, it follows that v cannot be one of the suspension vertices of S. At the same time, since vu_2 is not contained in any missing 2-face, u_2 must be one of the suspension vertices of S. This implies that $n = 4$ and $\mathrm{lk}(v, S) = C = (u_1, u_2, u_3, u_4, u_1)$.

On the other hand, since $g_2(\mathrm{lk}(u_2)) = 1$, the complex $\mathrm{lk}(u_2)$ is the join of C' with another cycle containing the path (u_1, v, u_3). The fact that $\partial\overline{u_1vu_3} \subseteq S \subseteq \Delta$ implies that $\mathrm{lk}(u_2) = C' * (u_1, v, u_3, u_1)$. Similarly, $\mathrm{lk}(u_4)$ contains the subcomplex $C' * (u_1, v, u_3)$ as well as the edge u_1u_3. Then, for $1 \leq i \leq 3$, we have $w_iu_1u_3 \in \mathrm{lk}(u_2)$, $w_iu_1u_4, w_iu_3u_4 \in \mathrm{lk}(v)$, and $u_1u_3u_4 \in S$. Hence $\partial(w_iu_1u_3u_4) \subseteq \Delta$. Since Δ has no missing 3-faces, it follows that $w_iu_1u_3u_4 \in \Delta$ for each

w_i. Thus, all 2-faces of $C' * (u_1, v, u_3, u_1)$ belong to $\mathrm{lk}(u_4)$. Because $\Delta \in S(2,4)$ and $\mathrm{lk}(u_4)$ is a sphere, we conclude that $\mathrm{lk}(u_4) = C' * (u_1, v, u_3, u_1)$. Consequently, $\Delta \supseteq \mathrm{st}(u_2) \cup \mathrm{st}(u_4) = C' * S$, which implies that $\Delta = C' * S$. This contradicts our assumption that $f_0(\Delta) > 8$. Therefore, the edge $e = vu_2$ must be contained in a missing 2-face. Since $\mathrm{lk}(v) = C * C'$, this missing face must necessarily be of the form vu_2u_j.

For the "furthermore" part, observe first that $w_1w_2w_3$ is a missing 2-face (otherwise $vw_1w_2w_3$ would be a missing 3-face of Δ). Also note that each of $\mathrm{lk}(vu_2)$, $\mathrm{lk}(vu_j)$, and $\mathrm{lk}(u_2u_j)$ contains $C' = (w_1, w_2, w_3, w_1)$, and so $C' * (v, u_2, u_j, v)$ is a *subcomplex* of Δ. (To justify that $\mathrm{lk}(u_2u_j)$ contains C', note that $\mathrm{lk}(u_2)$ is the join of a 3-cycle and some other cycle. Since, $\mathrm{lk}(u_2)$ contains C' and since $w_1w_2w_3$ is a missing face, we must have $\mathrm{lk}(u_2) = C' * D$ for some cycle D. Because $u_j \in \mathrm{lk}(u_2)$ but $u_j \notin C'$, it follows that $u_j \in D$, and hence $\mathrm{lk}(u_2u_j)$ contains C'.) Now, both vu_2u_j and $w_1w_2w_3$ are missing 2-faces. Therefore, $C' * (v, u_2, u_j, v)$ is an *induced* subcomplex of Δ. □

Lemma 4.4. *If $\mathrm{lk}(v)$ is the join of two 3-cycles, then every neighbor v' of v satisfies $g_2(\mathrm{lk}(v')) \geq 2$.*

Proof: Assume that $g_2(\mathrm{lk}(u_2)) = 1$. Then $\mathrm{lk}(u_2)$ is the join of C' with another cycle C'' containing the path (u_1, v, u_3). Since $u_1u_2u_3v$ is not a missing 3-face, it follows that $u_1u_2u_3$ must be a missing 2-face. Consequently, the cycle C'' has length at least 4. Now interchange the roles of v and u_2 and apply Lemma 4.3. It follows that the vertex $v \in \mathrm{lk}(u_2)$ must be adjacent to a vertex in $\mathrm{lk}(u_2)$ that lies outside the set $\{u_1, u_3\} \cup V(C')$. However, by assumption, $V(\mathrm{lk}(v)) = \{u_1, w_1, u_2, w_2, u_3, w_3\}$, and hence v has no such additional neighbor. This contradiction completes the proof. □

Lemma 4.5. *If $n > 3$, then every neighbor v' of v satisfies $g_2(\mathrm{lk}(v')) \geq 2$.*

Proof: We first consider neighbors $v' \in C$. Suppose, for contradiction, that $g_2(\mathrm{lk}(u_2)) = 1$. By Lemma 4.3, there exists a missing 2-face of the form vu_2u_j and $\Gamma = C' * (v, u_2, u_j, v)$ is an induced subcomplex of Δ. Write $\Delta = B_1 \cup B_2$, where B_1, B_2 are the two 4-balls whose common boundary is Γ. We claim that each B_i must contain at least two interior vertices. Otherwise, suppose $B_1 = t * \Gamma$. Then $g_2(\mathrm{lk}(t)) = 1$, and by Lemma 4.4, the neighbor u_2 of t must satisfy $g_2(\mathrm{lk}(u_2)) \geq 2$, a contradiction. The above argument shows that Δ is not reduced, which is again a contradiction. Therefore, each u_i satisfies $g_2(\mathrm{lk}(u_i)) \geq 2$.

Next, consider neighbors $w_i \in C'$. The link $\mathrm{lk}(w_i)$ contains the join of C and the path (w_{i+1}, v, w_{i-1}). Since $\Delta \in S(2,4)$, the triangle $w_1w_2w_3$ is a missing 2-face, so $\mathrm{lk}(w_i)$ cannot be the join of C and a 3-cycle. Consequently, $g_2(\mathrm{lk}(w_i)) \geq 2$. This completes the proof. □

Proof of Proposition 4.2: The statement follows from Lemmas 4.4 and 4.5. □

One consequence of this discussion is the following result.

Proposition 4.6. *If $\Delta \in S(2,4)$ is reduced and $f_0(\Delta) > 8$, then $g_2(\Delta) \geq \frac{2}{5} f_0(\Delta)$.*

Proof: Let I denote the set of vertices $v \in \Delta$ with $g_2(\mathrm{lk}(v)) = 1$. By Proposition 4.2, I forms an independent set. There are two cases to consider: either $|I| \geq 2f_0(\Delta)/5$ or $|I| \leq 2f_0(\Delta)/5$. In the former case, $g_2(\Delta) \geq |I| \geq 2f_0(\Delta)/5$ by Theorem 3.7. In the latter case, $g_2(\Delta) \geq (2f_0(\Delta)/5 + 2 \cdot 3f_0(\Delta)/5)/4 = 2f_0(\Delta)/5$ by McMullen's integral formula. In both cases, we conclude that $g_2(\Delta) \geq 2f_0(\Delta)/5$. □

The main result of this section now follows immediately from Propositions 4.1 and 4.6.

Theorem 4.7. *A sphere in $S(2,4)$ satisfies $g_2 \geq \frac{2}{5} f_0 - \frac{6}{5} = \frac{2}{5} g_1 + \frac{6}{5} g_0$.*

5 The g-numbers of spheres in $S(j, d-1)$

In this section, we study the g_i-numbers of spheres in $S(j, d-1)$ for general i and j. Our approach is inspired by a result of Chudnovsky and Nevo—Theorem 3.7—and its generalization in Proposition 3.6. In those results, one uses an independent set in the graph of Δ, together with Corollary 3.4, to construct a linearly independent collection of stresses on Δ. The main theorems of this section are obtained by further developing and applying this idea to spheres in $S(j, d-1)$.

Theorem 5.1. *Let $d \geq 5$ and let Δ be a flag normal $(d-1)$-pseudomanifold. Then the following holds:*

1. $g_2(\Delta) \geq (d-4)\alpha(\Delta)$. *This, in turn, implies that*

$$g_2(\Delta) \geq \frac{\sqrt{4d^2+12d-31}-(2d+1)}{4} f_0(\Delta).$$

2. *Furthermore, if $\Delta \in S(1, d-1)$, then $g_i(\Delta) \geq (d-2i)\alpha(\Delta)$ for all $2 \leq i \leq \frac{d-1}{2}$.*

Proof: Let $i \leq \frac{d-1}{2}$, and let I be a maximum independent set in the graph of Δ; that is, $|I| = \alpha(\Delta)$. Let $v \in I$ and let $F_1^v := v$. Since Δ is flag, the link of any face F is also flag. Consequently, if $\mathrm{lk}(F)$ is nonempty, then it contains two nonadjacent vertices. Specifically, choose nonadjacent vertices u_2, w_2 in $\mathrm{lk}(v)$, and let $F_2^v = vu_2$, $G_2^v = vw_2$ be the corresponding 1-faces of Δ. Continuing inductively, for $2 \leq k < d-2i-1$, assume that two $(k-1)$-faces $F_k^v = vu_2 \dots u_{k-1}u_k \in \Delta$ and $G_k^v = vu_2 \dots u_{k-1}w_k \in \Delta$ have already been defined. Then we choose a pair of nonadjacent vertices $u_{k+1}, w_{k+1} \in \mathrm{lk}(F_k^v)$, and define the corresponding k-faces $F_{k+1}^v = vu_2u_3 \dots u_ku_{k+1}$ and $G_{k+1}^v = vu_2u_3 \dots u_kw_{k+1}$. Finally, we define $G_1^v = vu_2u_3 \dots u_{d-2i}$.

First assume that $\Delta \in S(1, d-2)$. Choose a generic embedding of $V(\Delta)$ and, for $2 \leq k \leq d-2i$, apply Corollary 3.4 to the face G_k^v. Note that $\mathrm{lk}(G_k^v)$ is a flag sphere of dimension $d-k-1$ and that, by our choice of vertices u_j, w_j, the edge vw_k is contained in G_k^v, while for all $\ell > k$, the edge vw_ℓ is not contained in $\mathrm{st}(G_k^v)$. It follows from Corollary 3.4 that, for all $i \leq (d-k)/2$, there exists an affine i-stress ω_k^v on $\mathrm{st}(G_k^v)$ such that $vw_k \in \mathrm{supp}(\omega_k^v)$, while $vw_\ell \notin \mathrm{supp}(\omega_k^v)$ for all $\ell > k$. Likewise, there exists an affine i-stress ω_1^v on $\mathrm{st}(G_1^v)$ such that none of the edges $vw_2, \dots, vw_{d-2i}$ lie in the support of ω_1^v. Hence $\Omega^v := \{\omega_1^v, \dots, \omega_{d-2i}^v\}$ is a set of linearly independent stresses.

Now consider $\bigcup_{v \in I} \Omega^v$. The support of any stress in Ω^v contains v, but contains no other vertices of $I \backslash v$. Hence, the set of stresses $\bigcup_{v \in I} \Omega^v$ is also linearly independent. We conclude that

$$g_i(\Delta) \geq \Big|\bigcup_{v \in I} \Omega^v\Big| = (d-2i)|I| = (d-2i)\alpha(\Delta). \tag{5.1}$$

We now consider the case in which Δ is a flag normal pseudomanifold. In this setting, the link of every face of codimension ≥ 4 is itself a flag normal pseudomanifold with $g_2 \geq 1$ [34]. Together with Remark 3.5, this shows that when $i = 2$, the preceding argument applies equally well to flag normal pseudomanifolds. Consequently, the inequality $g_2(\Delta) \geq (d-4)\alpha(\Delta)$ continues to hold in this broader setting.

To prove the inequality on g_2 in terms of f_0, let $\epsilon = \frac{\sqrt{4d^2+12d-31}-(2d+1)}{4}$ and assume, for the sake of contradiction, that $g_2 < \epsilon f_0$. Then $f_1 = g_2 + df_0 - \binom{d+1}{2} < (d+\epsilon)f_0$. Hence, by Turán's theorem, there exists an independent set I with $|I| \geq \frac{f_0}{2(d+\epsilon)+1}$. Combining this with (5.1), yields

$$\epsilon f_0 > g_2 \geq \frac{(d-4)f_0}{2(d+\epsilon)+1},$$

which implies that $\epsilon > \frac{\sqrt{4d^2+12d-31}-(2d+1)}{4}$. This is the desired contradiction. $\square$

Remark 5.2. Since $4d^2+12d-31 = (2d+3)^2-22$, it follows that flag normal $(d-1)$-pseudomanifolds satisfy $g_2 \geq (1/2 - \delta(d))f_0$, where $\delta(d)$ is a function of d with $\delta(d) \to 0$ as $d \to \infty$.

Our next result relaxes the flagness assumption of Theorem 5.1(2).

Theorem 5.3. *Let $\Delta \in S(j, d-1)$, where $j = \min\{\lfloor \frac{d-1}{2} \rfloor - 1, d - 2i\}$. Then $g_i(\Delta) \geq 2\alpha(\Delta)$ for all $2 \leq i \leq d/3$.*

Proof: Consider a generic embedding of Δ in $\mathbb{R}^d$. Let I be an independent set of the graph of Δ of size $\alpha(\Delta)$, and fix $v \in I$. There are two possible cases to consider.

Case 1: The link $\mathrm{lk}(v)$ contains a missing $(r-1)$-face $F = u_1u_2 \dots u_r$ for some $2 \leq r \leq i$. Define the two $(r-1)$-faces $G_1 = vu_1 \dots u_{r-2}u_{r-1}$ and $G_2 = vu_1 \dots u_{r-2}u_r$. Since F is a missing face, $G_1 \notin \mathrm{st}(G_2)$ and $G_2 \notin \mathrm{st}(G_1)$. Next, observe that $i \leq \frac{d-i}{2} \leq \frac{d-r}{2}$. Because Δ has no missing faces of dimension $\geq d-2i+1$, it also has no missing faces of dimension $\geq d-r-i+1 \geq d-2i+1$. Therefore, $g_i(\mathrm{lk}(G_1)) \geq 1$ and $g_i(\mathrm{lk}(G_2)) \geq 1$. By Corollary 3.4, for each $k = 1, 2$, there exists an affine i-stress ω_v^k on $\mathrm{st}(G_k)$ whose support contains the face G_k. Since $G_1 \notin \mathrm{st}(G_2)$ and $G_2 \notin \mathrm{st}(G_1)$, we conclude that ω_v^1 and ω_v^2 are linearly independent.

Case 2: The link $\mathrm{lk}(v)$ has the complete $(i-1)$-skeleton. Since $\mathrm{lk}(v) \in S(j, d-2)$ with $j \leq \lfloor \frac{d-1}{2} \rfloor - 1$, it follows that $\mathrm{lk}(v)$ has at least $d+2$ vertices. (Indeed, every $(d-2)$-sphere with exactly $d+1$ vertices is of the form $\partial\sigma^\ell * \partial\sigma^{d-1-\ell}$ for some $1 \leq \ell \leq d-2$, and therefore has a missing face of dimension $\geq (d-1)/2$.) Let $W = \{w_1, \dots, w_{d+2}\}$ be any set of $d+2$ such vertices, and define $W_1 = W \backslash w_1$ and $W_2 = W \backslash w_2$. By our choice of generic embedding, for $k = 1, 2$, the affine dependence among the points $W_k \cup v$ yields an affine 1-stress $\delta_k = \sum_{u \in W_k \cup v} \nu_{u,k} x_u \in \mathcal{S}_1^a(\Delta)$, where $\nu_{u,k} \neq 0$ for all $u \in W_k \cup v$. Since $\mathrm{Skel}_{i-1}(\overline{W}) * v$ is a subcomplex of Δ, it follows that $\omega_v^1 := (\delta_1)^i$ and $\omega_v^2 := (\delta_2)^i$ are affine i-stresses with supports $\mathrm{Skel}_{i-1}(\overline{W_1 \cup v})$ and $\mathrm{Skel}_{i-1}(\overline{W_2 \cup v})$, respectively. These supports are distinct, and therefore ω_v^1 and ω_v^2 are linearly independent.

Finally, since I is an independent set, the collection $\bigcup_{v \in I}\{\omega_v^1, \omega_v^2\}$ of affine i-stresses is linearly independent. This gives $g_i(\Delta) \geq 2\alpha(\Delta)$, as claimed. $\square$

Theorem 5.1(1) provides a lower bound on g_2 in terms of f_0 for flag spheres. The next theorem generalizes this result, giving a lower bound on g_{k+1} in terms of f_{k-1} for certain values of $k > 1$, for spheres without large missing faces.

Theorem 5.4. *Let $k \geq 2$ and $d \geq 3(k+1)$. Then there exists a positive constant c_k, depending only on k, such that for every sphere $\Delta \in S(d-2-2k, d-1)$ with sufficiently many vertices,*

1. *$g_{k+1}(\Delta) \geq c_k \cdot \big(f_{k-1}(\Delta)\big)^{\frac{2k+2}{3k+1}}$ if $k \leq d/4$, and*
2. *$g_{k+1}(\Delta) \geq c_k \cdot \big(f_{k-1}(\Delta)\big)^{\frac{2k+2}{k+1+\lfloor d/2 \rfloor}}$ if $k \geq d/4$.*

Proof: We first treat the case $k \leq d/4$. Define the following graph K, which has $f_{k-1}(\Delta)$ vertices: each vertex corresponds to a $(k-1)$-face of Δ, and two vertices corresponding to $(k-1)$-faces σ and τ are connected by an edge if $\sigma \cup \tau$ is a face of Δ. In this case, $\dim(\sigma \cup \tau) = 2k - i - 1$ for some

$0 \le i \le k-1$. Since every set of size $2k-i$ can be expressed as the union of two of its k-subsets in $\frac{1}{2}\binom{2k-i}{k}\binom{k}{i}$ ways, it follows that the number of edges of K satisfies

$$f_1(K) = \sum_{i=0}^{k-1} \frac{1}{2}\binom{2k-i}{k}\binom{k}{i} f_{2k-i-1}(\Delta) < \frac{1}{2}\binom{2k}{k}\binom{k}{\lfloor k/2 \rfloor}\sum_{i=0}^{k-1} f_{2k-i-1}(\Delta).$$

We conclude that the average vertex degree of K is $\frac{2f_1(K)}{f_0(K)} < \frac{\binom{2k}{k}\binom{k}{\lfloor k/2\rfloor}\sum_{i=0}^{k-1} f_{2k-i-1}(\Delta)}{f_{k-1}(\Delta)}$. Hence, by Túran's theorem, the independence number $\alpha(K)$ of K satisfies

$$\alpha(K) > \frac{\big(f_{k-1}(\Delta)\big)^2}{\binom{2k}{k}\binom{k}{\lfloor k/2\rfloor}\sum_{i=0}^{k} f_{2k-i-1}(\Delta)}.$$

Consider a generic embedding of Δ in $\mathbb{R}^d$. If $\tau \in \Delta$ is a $(k-1)$-face, then $\mathrm{lk}(\tau) \in S(d-2-2k, d-1-k)$. In particular, $g_{k+1}(\mathrm{lk}(\tau)) \ge 1$. Let $m = \alpha(K)$ and let $\tau_1, \tau_2, \dots, \tau_m$ be $(k-1)$-faces of Δ representing a maximum independent set of K. Then for any $1 \le i \ne j \le m$, we have $\tau_i \notin \mathrm{st}(\tau_j)$ because $\tau_i \cup \tau_j \notin \Delta$. Hence, by Corollary 3.4, for each $1 \le i \le m$, there exists an affine $(k+1)$-stress $\omega_i \in \mathcal{S}^a_{k+1}(\mathrm{st}(\tau_i))$ such that $\tau_i \in \mathrm{supp}(\omega_i)$, but $\tau_j \notin \mathrm{supp}(\omega_i)$ for all $j \ne i$. Consequently, the stresses $\omega_1, \dots, \omega_m$ are linearly independent. We conclude that

$$g_{k+1}(\Delta) \ge m = \alpha(K) > \frac{\big(f_{k-1}(\Delta)\big)^2}{\binom{2k}{k}\binom{k}{\lfloor k/2\rfloor}\sum_{i=0}^{k} f_{2k-i-1}(\Delta)}.$$

For the remainder of the proof, all the g- and f-numbers refer to those of Δ. Write $g_{k+1} = \binom{x}{k+1}$ for some real number $x \ge k+1$. Assume that $g_{k+1} < c_k \cdot f_{k-1}^{\frac{2k+2}{3k+1}}$, where c_k is a positive constant to be determined later. Then

$$c_k \cdot f_{k-1}^{\frac{2k+2}{3k+1}} > g_{k+1} = \binom{x}{k+1} > \frac{(x-k)^{k+1}}{(k+1)!}.$$

That is, $x - k < ((k+1)!c_k)^{\frac{1}{k+1}} \cdot \big(f_{k-1}(\Delta)\big)^{\frac{2}{3k+1}}$. Using the g-theorem, along with the fact that for $1 \le s \le k$ and $x \ge k+1$, $x+s-1 \le (k+s)(x-k)$, we infer that for all $1 \le s \le k$,

$$g_{k+s} \le \binom{x+s-1}{k+s} < \frac{(x+s-1)^{k+s}}{(k+s)!} \le \frac{(k+s)^{k+s}(x-k)^{k+s}}{(k+s)!} \le c_{k,s} \cdot f_{k-1}^{\frac{2(k+s)}{3k+1}},$$

where $c_{k,s} = \frac{(k+s)^{k+s}}{(k+s)!}((k+1)!c_k)^{\frac{k+s}{k+1}}$.

Since $\Delta \in S(d-2-2k, d-1)$, we have $g_{2k} \ge 1$. From the defining relations for the g-numbers, each f_{2k-i-1} can be expressed as a linear combination of $g_{2k}, g_{2k-1}, \dots, g_{k+2}, g_{k+1}, f_{k-1}, \dots, f_0, 1$. Moreover, when $i = 0$, the coefficient of g_{2k} in this linear combination is 1, while for $i > 0$ this coefficient is zero. By Björner's result [5, Theorem 5], the first half of the f-numbers of Δ is weakly increasing. Thus, for $k \le d/4$, $f_0 \le f_1 \le \dots \le f_{k-1}$. Also the exponents $\{2(k+s)/(3k+1)\}_{s=1}^{k}$ form an increasing sequence, whose largest term $4k/(3k+1) > 1$. Therefore, letting $c'_k = c_{k,k}$, we obtain that for $f_0 \gg 0$,

$$\sum_{i=0}^{k} f_{2k-i-1} < 2c'_k \cdot f_{k-1}^{\frac{4k}{3k+1}}.$$

Putting everything together, we conclude that

$$c_k \cdot f_{k-1}^{\frac{2k+2}{3k+1}} > g_{k+1} > \frac{f_{k-1}^2}{\binom{2k}{k}\binom{k}{\lfloor k/2 \rfloor}\sum_{i=0}^{k} f_{2k-i-1}} > \frac{f_{k-1}^{\frac{2k+2}{3k+1}}}{2\binom{2k}{k}\binom{k}{\lfloor k/2 \rfloor} \cdot c'_k}.$$

Choosing $c_k > 0$ so that $c_k \cdot c'_k \cdot 2\binom{2k}{k}\binom{k}{\lfloor k/2 \rfloor} < 1$ leads to a contradiction. Thus, for such a choice of c_k, $g_{k+1} \geq c_k \cdot f_{k-1}^{\frac{2k+2}{3k+1}}$.

The treatment of the case $\frac{d}{4} \leq k \leq \frac{d}{3} - 1$ is very similar. The only difference is that when $k \geq d/4$, each of $f_k, f_{k+1}, \ldots, f_{2k-1}$ can be written as a linear combination of

$$g_{\lfloor d/2 \rfloor}, \ldots, g_{k+2}, g_{k+1}, f_{k-1}, \ldots, f_0, 1.$$

Since the inequalities $g_{k+s} \leq O\left((g_{k+1})^{(k+s)/(k+1)}\right)$ continue to hold for all $1 \leq s \leq \lfloor d/2 \rfloor - k$, and since $g_{\lfloor d/2 \rfloor} > 0$ for $\Delta \in S(d-2-2k, d-1)$, computations analogous to those in the previous case apply. These yield the bound stated in the theorem. □

6 Level rings and counterexamples to conjectures on stresses

In this section, we switch gears and use algebraic tools to obtain additional relations among the g-numbers of spheres in $S(j, d-1)$.

We begin by reviewing some basics on Gorenstein and level rings. Let $R = \mathbb{R}[x_1, \ldots, x_n]$, let $\mathbf{m} = (x_1, \ldots, x_n)$ be the irrelevant ideal (that is, the maximal homogeneous ideal) of R, and let N be a graded R-module. One example of such N is $\mathbb{R}[\Delta] = R/I_\Delta$. In fact, all of the R-modules we consider are quotients of $\mathbb{R}[\Delta]$.

Define the *socle* of N by $\mathrm{soc}^R N = 0 :_N \mathbf{m} = \{y \in N : \mathbf{m}y = 0\}$ and let $r_k(N) := \dim_{\mathbb{R}}(\mathrm{soc}^R N)_k$. If N has Krull dimension 0, then the largest k for which $r_k(N) \neq 0$ is called the *socle degree of* N; we denote it by s. The integer vector $S(N) = (r_1(N), \ldots, r_s(N))$ is called the *socle vector of* N.

If $N = \mathbb{R} \oplus N_1 \oplus \cdots \oplus N_s$ (with $N_s \neq 0$) as an $\mathbb{R}$-vector space, then $(\mathrm{soc}^R N)_s = N_s$, and hence the socle degree of N is s. We say that N is *level* if its socle vector $S(N)$ is of the form $(0, \ldots, 0, a)$ for some $a \geq 1$, and *Gorenstein* if in addition $a = 1$. A sequence is called a *level sequence* (resp. *Gorenstein sequence*) if it is the the Hilbert function of an Artinian level ring (resp. Artinian Gorenstein ring) N; that is, it is of the form $(1, \dim_{\mathbb{R}} N_1, \ldots, \dim_{\mathbb{R}} N_s)$. In particular, every level sequence is an M-sequence.

Assume that (Δ, p) is a $(d-1)$-sphere with n vertices and with an embedding $p : V(\Delta) \to \mathbb{R}^d$ that is either generic or natural (when Δ is polytopal). As in Section 2, let $\Theta = \Theta(p)$ be the l.s.o.p. of $\mathbb{R}[\Delta]$ associated with p and let $c = \sum_{v=1}^{n} x_v$. Recall that $\mathbb{R}[\Delta]/(\Theta)$ is Gorenstein. On the other hand, if we set $A(\Delta) := \mathbb{R}[\Delta]/(\Theta, c)$, then by the g-theorem, we have $\dim_{\mathbb{R}} A(\Delta)_k = g_k$ for $0 \leq k \leq \lfloor d/2 \rfloor$ and $\dim_{\mathbb{R}} A(\Delta)_k = 0$ otherwise. So the socle degree of $A(\Delta)$ is at most $\lfloor d/2 \rfloor$.

The relevance of level rings to spheres with no large missing faces is explained by the following result; see [25, Proposition 3.2].

Proposition 6.1. *Let (Δ, p) be a $(d-1)$-sphere with n vertices and with a generic or natural embedding p in $\mathbb{R}^d$. Let $\Theta = \Theta(p)$ be the l.s.o.p. associated with p. Denote by m_i the number of missing i-faces of Δ. Then*

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{soc}^R A(\Delta)_k &= m_{d-k} \quad \text{if } k < \lfloor (d-1)/2 \rfloor, \text{ and} \\ \dim_{\mathbb{R}} \mathrm{soc}^R A(\Delta)_k &\geq m_{d-k} \quad \text{if } k = \lfloor (d-1)/2 \rfloor. \end{aligned} \tag{6.1}$$

An immediate, albeit rather surprising, consequence of this result and the definition of level rings is the following.

Corollary 6.2. *Let Δ be a $(d-1)$-sphere, let u be the largest integer such that $\Delta \in S(d-u, d-1)$, and set $\tilde{u} := \min\{u, \lfloor (d-1)/2 \rfloor\}$. Then, under the assumptions and notation of Proposition 6.1, the graded algebra $\oplus_{i=0}^{\tilde{u}} A(\Delta)_i$ is a level ring of socle degree $\tilde{u}$. In particular, $(g_0(\Delta), g_1(\Delta), \ldots, g_{\tilde{u}}(\Delta))$ is a level sequence.*

Proof: Since $\Delta \in S(d-u, d-1)$, Proposition 6.1 implies that $A(\Delta)$ has vanishing socle in all degrees $< \tilde{u}$. Therefore, the truncated algebra $\oplus_{i=0}^{\tilde{u}} A(\Delta)_i$ is a level ring of socle degree $\tilde{u}$. □

Level sequences have been extensively studied, and, as the next two theorems demonstrate, levelness imposes far stronger restrictions on the entries of a sequence than merely being an M-sequence. The following theorem summarizes several results from [13, Theorem 2.2] and [32, Section III.3].

Theorem 6.3. *Let $\ell = (1, \ell_1, \ldots, \ell_s)$ be a level sequence. Then:*

1. *For every $1 \leq s' \leq s$, the truncated sequence $(1, \ell_1, \ldots, \ell_{s'})$ is also level.*
2. *For all $i, j \geq 1$ with $i + j \leq s$, one has $\ell_i \leq \ell_j \ell_{i+j}$.*
3. *The reverse sequence $(\ell_s, \ldots, \ell_1, 1)$ is the sum of a Gorenstein sequence $(1, b_1, \ldots, b_{s-1}, 1)$ and of $\ell_s - 1$ additional M-sequences (each ending in a zero). In particular, if $\ell_s = 1$, then ℓ is Gorenstein, and hence symmetric.*

In view of part 3 of this theorem, it is natural to look for structural properties satisfied by sums of M-sequences. Several such properties are collected in the following theorem; see [20, Proposition 4.3]. This theorem follows from a stronger result [6, 16], which extends Macaulay's theorem to graded modules.

Theorem 6.4. *Let $\ell = (\ell_0, \ldots, \ell_s)$ be a sequence of nonnegative integers. The following statements are equivalent:*

1. *The sequence ℓ is a sum of M-sequences.*
2. *Either $\ell_i = 0$ for all i, or $\ell_0 \geq 1$ and the sequence $(1, \ell_1, \ldots, \ell_s)$ is an M-sequence.*
3. *The sequence ℓ is the Hilbert function of a graded module over some polynomial ring $\mathbb{R}[x_1, \ldots, x_m]$ generated in degree 0.*

Putting Corollary 6.2 and Theorems 6.3 and 6.4 together yields the following restrictions on the g-numbers of spheres without large missing faces. These relations constitute the main result of this section.

Corollary 6.5. *Let Δ be a $(d-1)$-sphere, let u be the largest integer such that $\Delta \in S(d-u, d-1)$, and set $\tilde{u} := \min\{u, \lfloor (d-1)/2 \rfloor\}$. Then the sequence $(g_0(\Delta), g_1(\Delta), \ldots, g_{\tilde{u}}(\Delta))$ is a level sequence. In particular,*

1. *For all $i, j \geq 1$ with $i + j \leq \tilde{u}$, one has $g_i \leq g_j g_{i+j}$.*
2. *Both $(1, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ and $(1, g_{\tilde{u}-1}, \ldots, g_1, 1)$ are M-sequences.*

The following special case of Corollary 6.5 deserves separate mention:

Corollary 6.6. *Let $\Delta \in S(u+1, 2u)$. Then $g(\Delta) = (g_0(\Delta), g_1(\Delta), \dots, g_u(\Delta))$ is a level sequence. In particular, if $g_u(\Delta) = 1$, then the g-vector of Δ is a Gorenstein sequence and hence symmetric.*

In light of Corollary 6.6, one may wonder whether the g-vectors of spheres in $S(u, 2u-1)$ are also level. To investigate this question, recall the following conjecture on the structure of the spaces of affine stresses, proposed in [28, Conjecture 3.6].

Conjecture 6.7. *Assume $u \le d/2$. Let (Δ, p) be a $(d-1)$-sphere in $S(d-u, d-1)$ with a generic or natural embedding p in $\mathbb{R}^d$. Then for all $1 \le j < i \le u$,*

$$\mathcal{S}^a_j(\Delta, p) = \{\partial_\mu \omega : \omega \in \mathcal{S}^a_i(\Delta, p), \mu \in R_{i-j}\}. \tag{6.2}$$

This conjecture was proved in [25, Theorem 3.1] for all $u \le \lceil \frac{d}{2} \rceil - 1$. The case $u = d/2$ was left open. More precisely, when $d = 2u$, [25, Theorem 3.1] established that (6.2) holds for all $1 \le j < i \le u-1$; moreover, it showed that the validity of (6.2) for $j = u-1$, $i = u$ is equivalent to the inequality of (6.1) holding as equality. In other words, when $d = 2u$,

$$\mathcal{S}^a_{u-1}(\Delta, p) = \{\partial_{x_v} \omega : \omega \in \mathcal{S}^a_u(\Delta, p), v \in [n]\} \quad \text{if and only if} \quad \dim_{\mathbb{R}} \operatorname{soc}^R A(\Delta)_{u-1} = m_{2u-(u-1)} = 0.$$

That is, Conjecture 6.7 holds for $d = 2u$ and $\Delta \in S(u, 2u-1)$ if and only if $A(\Delta)$ is a level ring.

Our next task in this section is to construct, for all $u \ge 3$, a sphere in $S(u, 2u-1)$ whose g-vector is not level, thereby showing that Conjecture 6.7 is false when $d = 2u \ge 6$. Meanwhile, it is worth noting that Conjecture 6.7 does hold when $u = 2$, $\Delta \in S(2,3)$, and p is generic; this was proved in [9, Theorem 8.4]. The case where p is natural remains open. We also mention that Conjecture 6.7 holds when $d = 2u$ and Δ is a flag PL $(d-1)$-sphere; see [28, Theorem 6.3]. In particular, $A(\Delta)$ is level in these cases.

Example 6.8. Let $u \ge 3k \ge 3$, and consider the sphere

$$K(u-k, 2u-1) = \partial\sigma^{u-k} * \partial\sigma^{u-k} * \partial\sigma^{2k} \in S(u-k, 2u-1).$$

Since $K(u-k, 2u-1)$ is a join of boundaries of simplices, we have

$$h(K(u-k, 2u-1), t) = h(\partial\sigma^{u-k}, t)^2 h(\partial\sigma^{2k}, t) = (1 + t + \cdots + t^{u-k})^2 (1 + t + \cdots + t^{2k}).$$

When $j \le 2k$, the coefficient $h_j(K(u-k, 2u-1))$ counts the number of weak compositions of j into three parts. Consequently, $g_j(K(u-k, 2u-1)) = \binom{j+2}{2} - \binom{j+1}{2} = j+1$. Similarly, when $2k \le j \le u-k$, the value $h_j(K(u-k, 2u-1))$ is the sum, over $0 \le \ell \le 2k$, of the numbers of weak compositions of $j - \ell$ into two parts. Thus, in this range, $h_j(K(u-k, 2u-1)) = \sum_{\ell=0}^{2k} (j - \ell + 1)$, and hence $g_j(K(u-k, 2u-1)) = \sum_{\ell=0}^{2k} 1 = 2k+1$ for all $2k+1 \le j \le u-k$. Finally, when $u-k \le j \le u$, $h_j(K(u-k, 2u-1)) = \sum_{\ell=0}^{2k} h_{j-\ell}(\partial\sigma^{u-k} * \partial\sigma^{u-k})$. Therefore, for $u-k < j \le u$,

$$g_j(K(u-k, 2u-1)) = (h_j - h_{j-2k-1})(\partial\sigma^{u-k} * \partial\sigma^{u-k}) = (2(u-k) + 1 - j) - (j - 2k) = 2u + 1 - j.$$

To summarize,

$$g_j(K(u-k, 2u-1)) = \begin{cases} j+1 & \text{if } 0 \le j \le 2k \\ 2k+1 & \text{if } 2k+1 \le j \le u-k \\ 2u+1-2j & \text{if } u-k+1 \le j \le u \end{cases}.$$

Corollary 6.9. *Conjecture 6.7 does not hold when $d = 2u \geq 6$: for all $u \geq 3$, there exists a polytopal sphere $\Delta \in S(u, 2u-1)$ such that, regardless of whether p is generic or natural, the ring $A(\Delta)$ is not level.*

Proof: By the discussion following Conjecture 6.7, if $\Delta \in S(u, 2u-1)$ satisfies the conjecture, then the g-vector of Δ must be a level sequence. In particular, if $g_u(\Delta) = 1$, then $g(\Delta)$ must be Gorenstein, and hence symmetric. Now consider $K(u-k, 2u-1)$ with $k \geq 1$ and $u \geq 3k$. The computations carried out in Example 6.8 show that while $g_u(K(u-k, 2u-1)) = 1$, one has $g_1(K(u-k, 2u-1)) = 2$ and $g_{u-1}(K(u-k, 2u-1)) = 3$, so the required symmetry fails. Thus, the complexes $K(u-k, 2u-1) \in S(u-k, 2u-1) \subseteq S(u, 2u-1)$ violate the conjecture. □

We recall another conjecture—this one is on the support of affine stresses—proposed in [28, Conjecture 3.3], which we are now able to disprove.

Conjecture 6.10. *Let $2 \leq i \leq d/2$, let $\Delta \in S(d-i, d-1)$, and let p be a generic or natural embedding of Δ in $\mathbb{R}^d$. Then every $(i-1)$-face of Δ participates in some affine i-stress on (Δ, p).*

While Conjecture 6.10 holds for $i = 2$ (see [36] for the case of generic embeddings and [25, Theorem 1.5] for the case of natural embeddings), for following result shows that it fails in general.

Corollary 6.11. *For any $m \geq 1$, there exists a simplicial $(4m+2)$-polytope P with $\partial P \in S(2m, 4m+1)$, and a $2m$-face of P that does not participate in any affine $(2m+1)$-stress on $(\partial P, p)$, where p is the natural embedding of ∂P.*

Proof: Let $u = 2m+1$ and consider the sphere $K(2m, 4m+1) = \partial\sigma^{2m} * \partial\sigma^{2m} * \partial\sigma^2 \in S(2m, 4m+1)$ from Example 6.8. This sphere can be realized as the boundary of a simplicial $(4m+2)$-polytope P on vertex set $[2u+3]$, where the position vectors of the vertices $1, 2, \dots, 2u+3$ are given by

$$e_1, e_2, \dots, e_{u-1}, -\sum_{i=1}^{u-1} e_i, \quad e_u, e_{u+1}, \dots, e_{2u-2}, -\sum_{i=u}^{2u-2} e_i, \quad e_{2u-1}, e_{2u}, -(e_{2u-1}+e_{2u}).$$

Let $y_1 = x_1 + x_2 + \cdots + x_u$, $y_2 = x_{u+1} + x_{u+2} + \cdots + x_{2u}$, and $y_3 = x_{2u+1} + x_{2u+2} + x_{2u+3}$. Then the definition of p implies that $y_1 - \frac{u}{3} \cdot y_3$ and $y_2 - \frac{u}{3} \cdot y_3$ belong to $\mathcal{S}^a_1(\partial P, p)$. Since the only missing faces of ∂P are $\{1, 2, \dots, u\}$, $\{u+1, u+2, \dots, 2u\}$, and $\{2u+1, 2u+2, 2u+3\}$, we conclude that $f = (y_1 - \frac{u}{3} \cdot y_3)(y_2 - \frac{u}{3} \cdot y_3)(y_1 - y_2)^{u-2}$ is an element of $\mathcal{S}^a_u(\partial P, p)$. Furthermore, since $g_u(\partial P) = 1$, the stress f forms a basis of $\mathcal{S}^a_u(\partial P, p)$. Now, the coefficient of $y_1^m y_2^m y_3$ in f is

$$-\frac{2m+1}{3} \cdot \binom{2m-1}{m}(-1)^{m-1} - \frac{2m+1}{3} \cdot \binom{2m-1}{m-1}(-1)^m = 0.$$

In particular, no $2m$-face of ∂P the form $F \cup G \cup v$, where F is an m-subset of $\{1, 2, \dots, u\}$, G is an m-subset of $\{u+1, u+2, \dots, 2u\}$, and $v \in \{2u+1, 2u+2, 2u+3\}$, is in the support of f. Therefore, no face of this form is in the support of any element of $\mathcal{S}^a_u(\partial P, p)$, and so the conjecture fails in this case. □

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